Photo-multiplier and cable analysis

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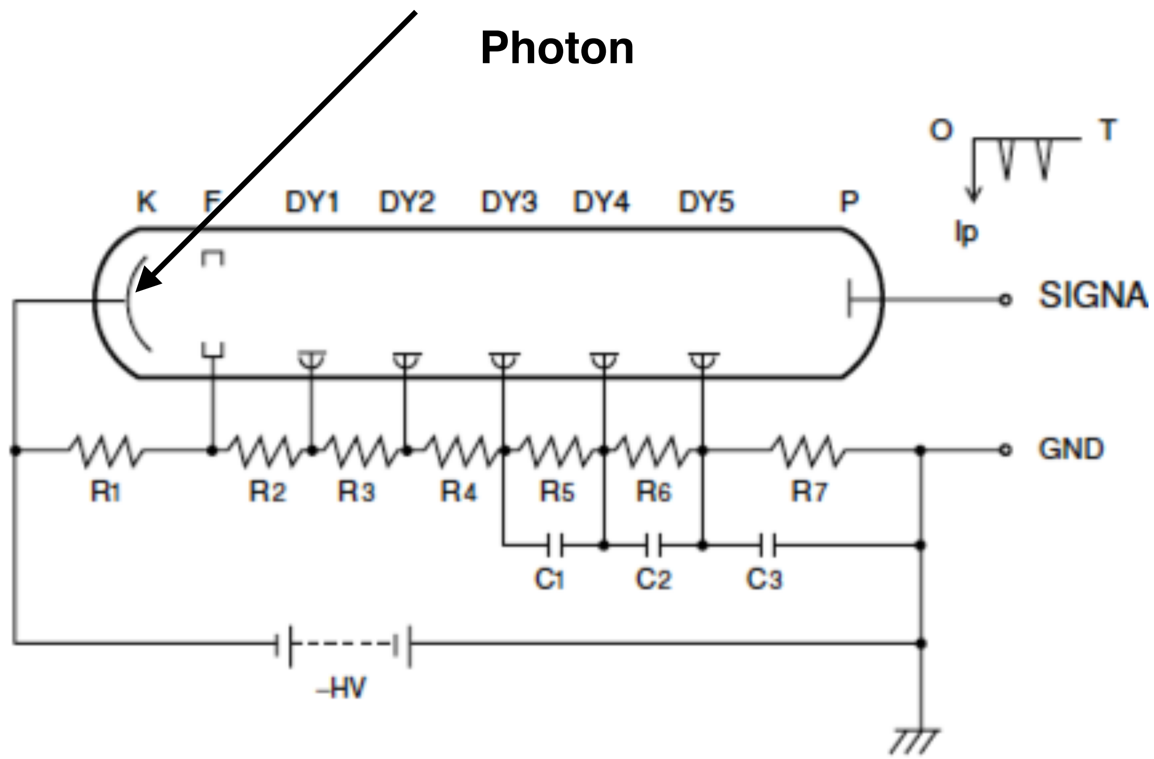
January 8, 2019

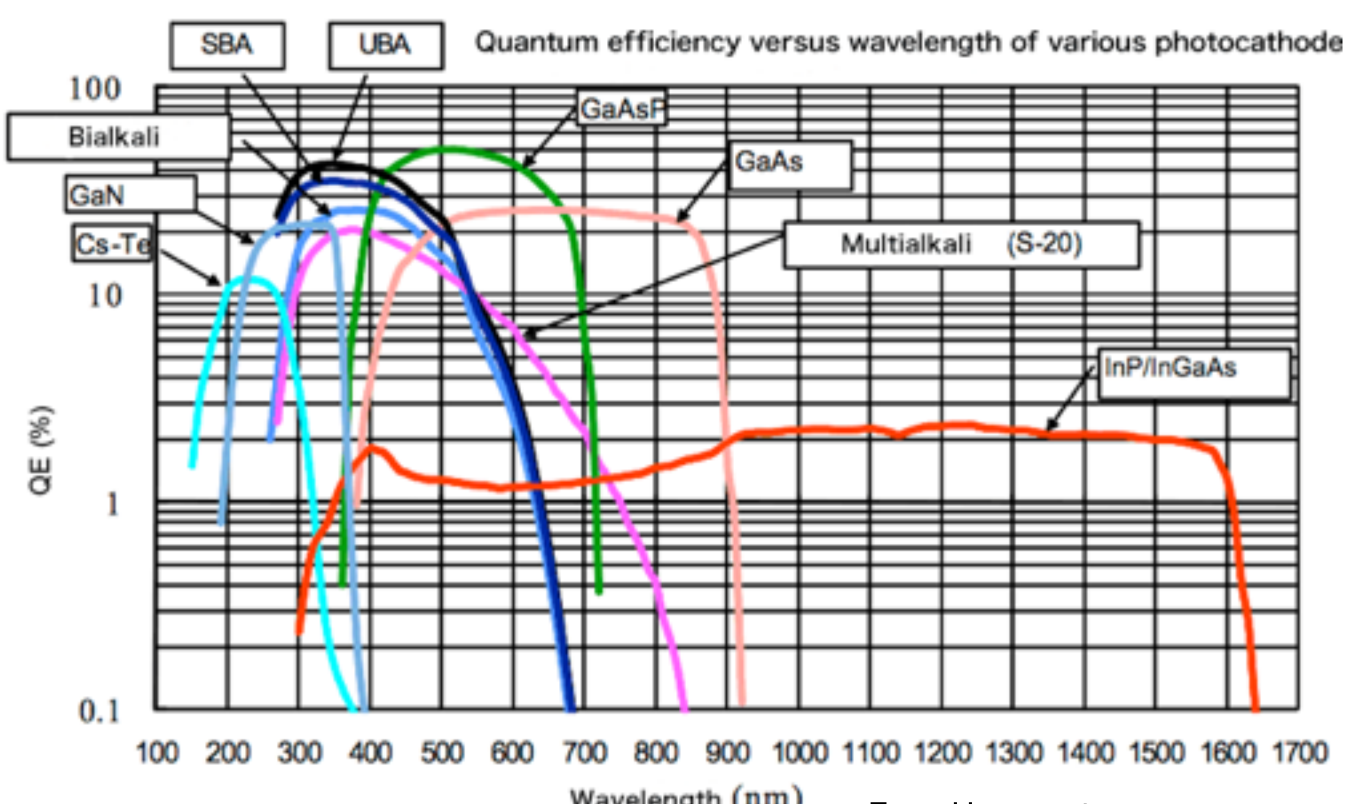
Updated April 26, 2019

Purpose

- ***One of the most common detector configurations is a photo-multiplier tube attached to a coaxial cable.***
- ***The signal is integrated or digitized at the end of a long cable.***
- ***These set of notes will provide basic understanding of how to analyze the signal and propagate it to the end of the cable.***
- ***Much of this material can be obtained from classic text books on detector instrumentation such as Knoll (2012).***
- ***In these set of notes, the attempt is to be brief and provide a complete case study and give some instinct. I also provide some guidance to understand pathologies that are common in these systems.***
- ***These notes come together with a spread sheet and a mathematica notebook. These can be used to perform many of the calculations here.***

Photo-Multiplier Tube





From Hamamatsu

- Photons are converted to charge by a photocathode with low work function.
- Electric fields accelerate and multiply the primary electron in several stages. Each stage has multiplication of ~4-5.
- Typical Gain = $AV^{kn} \sim 10^6$- 10^7 where V is the typical voltage ~ few 1000 V.
- Time resolution < 10 ns.
- Transit time can be <1 microsec
- PMT first stage is sensitive to small magnetic fields.
- Many clever geometries.

Photomultiplier (PMT) basics

Electrotr ode	Nominal Ratio	Resistor	Voltage	dampng or load resistor	stabiliza tion capacito	Voltage drop	Estimated Gain	Power Watt	capacitor charge C
K			-2000						
	16.8	1650000				619.1369606	7	-0.232321561	
Dy1			-1380.863039						
	4	540000				202.6266417	5.733232362	-0.076032511	
Dy2			-1178.236398						
	5	780000				292.6829268	6.35809185	-0.109824738	
DY3			-885.5534709						
	3.33	440000				165.1031895	5.123567144	-0.061952416	
DY4			-720.4502814						
	1.67	270000				101.3133208	3.615571693	-0.038016255	
dy5			-619.1369606						
	1	150000				56.28517824	2.193578227	-0.021120142	
dy6			-562.8517824						
	1.2	180000				67.54221388	2.576815687	-0.02534417	
dy7			-495.3095685	0					
	1.5	220000				82.55159475	3.059032499	-0.030976208	
dy8			-412.7579737	100					
	2.2	330000			0	123.8273921	4.215612053	-0.046464312	0
dy9			-288.9305816	100					
	3	440000			0.000000(	165.1031895	5.123567144	-0.061952416	7.26454E-06
dy10			-123.8273921	100					
	3.4	330000			0.000000(	123.8273921	4.215612053	-0.046464312	5.44841E-06
P			0	open					
SUM		5330000					7441972.676	0.75 W	
Current		-0.000375235	amp						

Example of a base for Hamamatsu R5912 10 stage tube.

Considerations

Max voltage.

Max voltage drop from K to the first dynode determines the first stage gain.

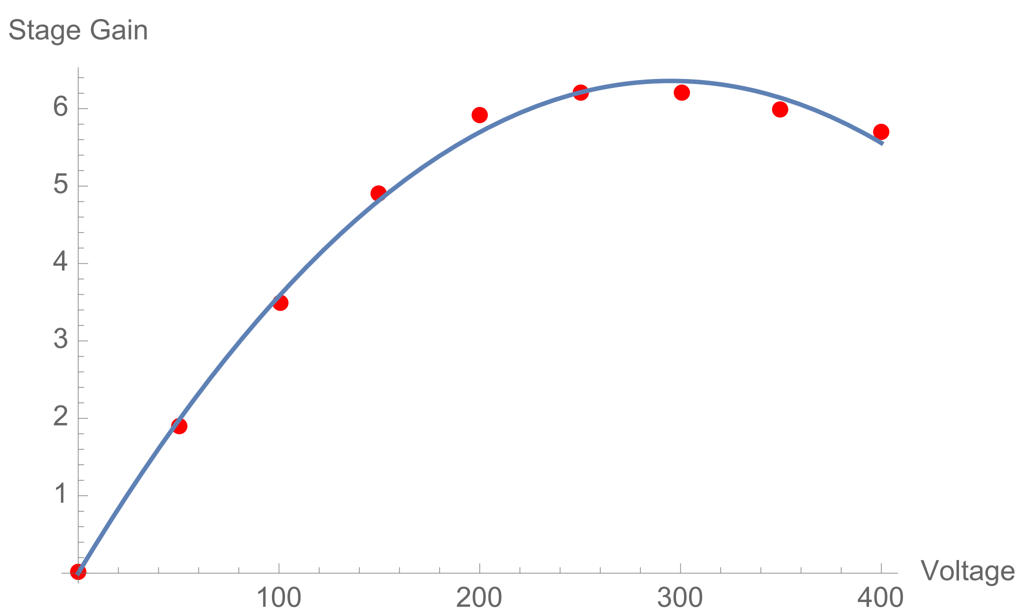
Total current should be high enough compared to expected current through tube.

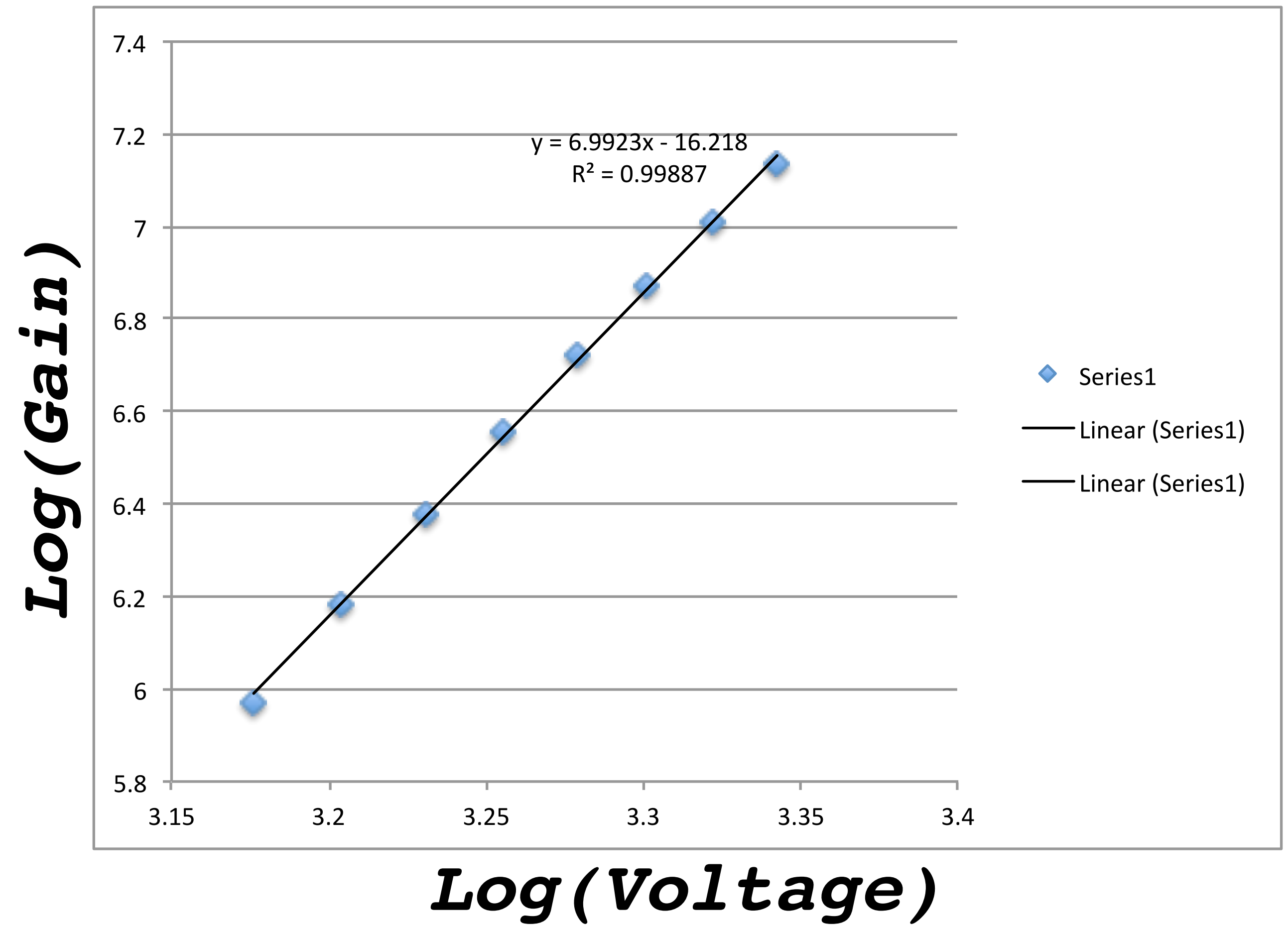
The charge in the stabilization capacitors should be larger than expected charge from a signal pulse.

If power is too high it can heat up the base.

Gain at each stage is calculated using $g(V) = 0.04*V-0.0007*V^2$

Basic calculation of gain

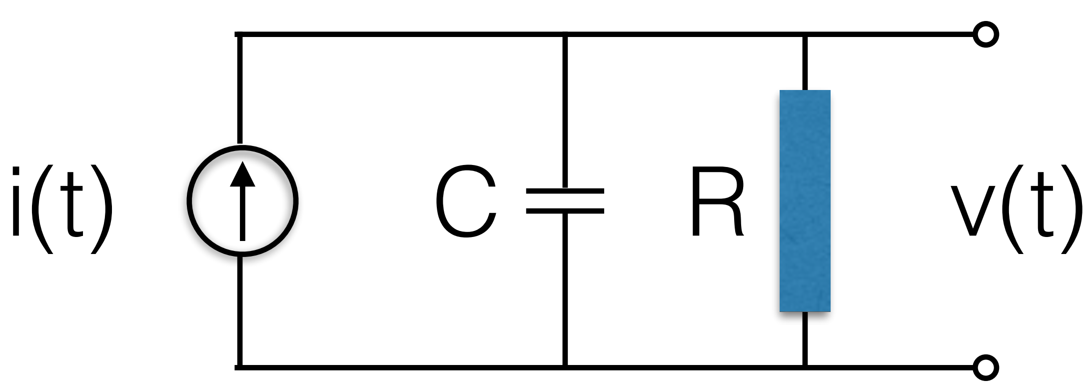




- ***For the previous calculation a model for gain was used from a handbook. The model is different for each type of photo-cathode material. This one is for bialkali.***
- ***This resulted in a voltage versus gain curve on the right. The slope of the log/log curve is the exponent of the voltage n= ~7.0***
- ***This also means that if voltage changes by 1% => gain changes by 7 %***

PMT equivalent circuit

- ***The circuit on slide 3 for the PMT base looks complicated. How is it to be modeled ? Surely the shape of the pulse depends on all those resisters and capacitors ?***
- ***Luckily there is Norton's theorem for passive circuits: it states that no matter how complicated a network the equivalent circuit is a current source in parallel with a resistor (for DC circuits) or impedance for AC circuits.***
- ***In the case of a PMT it would be a resistor and a capacitor. The values of these can be calculated from the network, but we don't have to. We can just measure them.***
- ***For the current source we can assume it is a delta function for a single electron pulse; but it is better to assume that the current also has an exponential shape to account for scintillation lifetime.***



PMT equivalent

We assume that there are no inductances. And the current produced at the anode has an exponetial fall.

$$i(t) = \frac{q_0}{\tau_s} e^{-t/\tau_s} \text{ for } t > 0, \text{ and } 0 \text{ for } t < 0$$

Recall that R could represent the coaxial cable; if it has impedance 50 Ω then R = 50 Ω. (we will work on this later). C represents various capacitances on the base. Set $\tau = RC$

$$i(t) = C\frac{dv(t)}{dt} + \frac{v(t)}{R}$$

PMT signal solution

$$i(t) = \frac{q_0}{\tau_s} e^{-t/\tau_s} \text{ for } t > 0, \text{ and } 0 \text{ for } t < 0$$

Set $\tau = RC$

There are two ways to solve this. First we do the differential equation.

$$i(t) = C\frac{dv(t)}{dt} + \frac{v(t)}{R}$$

$$v(t) = -\frac{q_0 R}{(\tau - \tau_s)} \times \left(e^{-t/\tau_s} - e^{-t/\tau}\right) \text{ for } \tau \neq \tau \text{ and } t > 0$$

$$v(t) = \frac{q_0 R}{\tau_s^2} t \times e^{-t/\tau_s} \text{ for } \tau = \tau_s$$

Second way is by Fourier transform. This will allow much more flexibility in the long run. We replace the differential equation by its Fourier transform.

$$i(t) \Rightarrow I(\omega) = \frac{q_0}{\tau_s} \frac{\tau_s}{(1 + i\omega\tau_s)}$$

Recall that the impedance of C and R in parallel is $\dfrac{R}{(1 + i\omega RC)}$

$$I(\omega) = V(\omega) \times \frac{(i\omega\tau + 1)}{R} \Rightarrow V(\omega) = q_0 R \times \frac{1}{(1 + i\omega\tau_s)(1 + i\omega\tau)}$$

PMT signal solution by Fourier transform

We need to take inverse Fourier transform of

$$V(\omega) = \frac{q_0 R}{(1+i\omega\tau_s)(1+i\omega\tau)}$$

use partial fraction separation first.

$$V(\omega) = -\frac{q_0 R}{(\tau-\tau_s)}\left[\frac{\tau_s}{(1+i\omega\tau_s)} - \frac{\tau}{(1+i\omega\tau)}\right]$$

by inspection we get

$$v(t) \ = -\frac{q_0 R}{(\tau-\tau_s)}\left(e^{-t/\tau_s} - e^{-t/\tau}\right)u(t)$$ where $u(t)$ is the unit step function.

if $\tau = \tau_s$ then

$$V(\omega) = \frac{q_0 R}{(1+i\omega\tau)^2} = \frac{q_0 R}{\tau^2}\frac{1}{(1/\tau + i\omega)^2}$$

by inspection (use standard tables of Fourier transforms)

$$v(t) = -\frac{q_0 R}{\tau^2} t \times e^{-t/\tau}$$

example PMT signal plots

Here we will plot what the PMT pulse looks like. First some numbers: Recall that q_0 is the charge at the anode and so it includes the gain from the PMT.

$q_0 = GNe$ where G is the gain and Ne is the charge at the cathode.

Assume $G = 10^7$ and $N = 1 \Rightarrow q_0 = 1.6 \times 10^{-12} C = 1.6 pC$

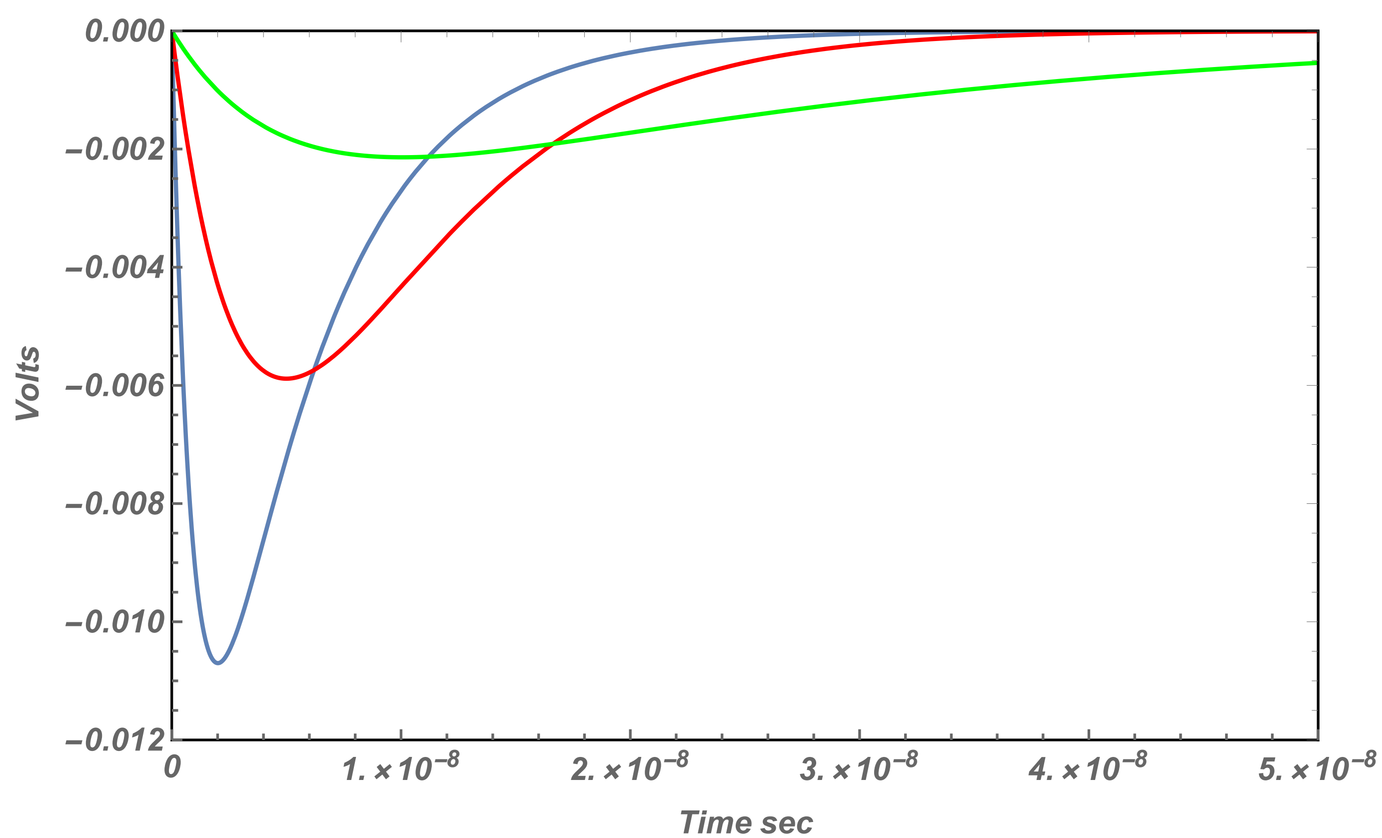
We now plot 3 cases

$\tau_s < \tau$: set $\tau_s = 1ns$ and τ=5 ns

$RC = 5ns \Rightarrow C = 0.1nF$ for R = 50 Ω

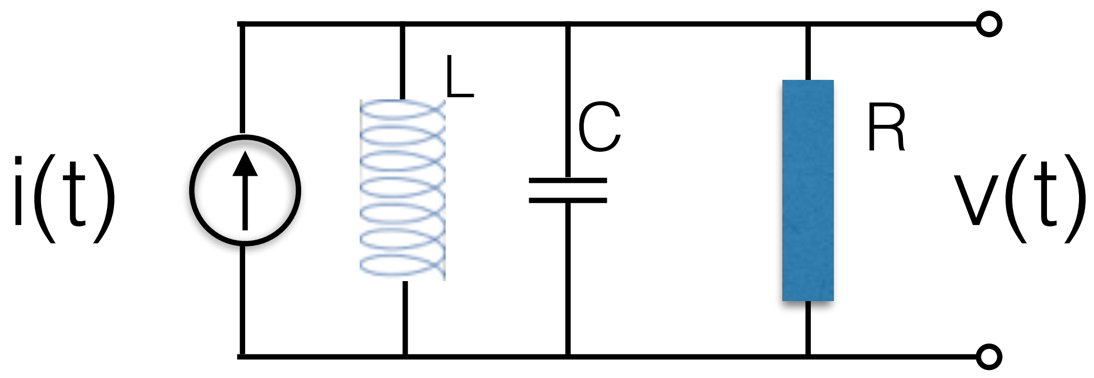
$\tau_s = \tau = 5ns$

$\tau_s > \tau$: set $\tau_s = 25ns$ and τ=5 ns



Rise time is the shorter time and fall time is the longer time

pmt pulse with inductances



What happens if there is some inductance at the anode of a PMT. This could happen due to bad cabling. Inductances might be difficult to avoid.

If the inductance is in parallel then the DC component is shorted out. The integral of the output pulse must be 0.

Recall that the impedance of L, C and R in parallel is

$$Z = \frac{1}{(1/R + i\omega C + 1/i\omega L)}$$

$$I(\omega) = V(\omega)/Z \Rightarrow V(\omega) = \frac{q_0}{(1+i\omega\tau_s)} \times \frac{i\omega R}{(i\omega + R/L - \omega^2 RC)}$$

Set $\tau = RC$ and $\omega_0^2 = 1/LC$

$$V(\omega) = \frac{q_0 R}{(1+i\omega\tau_s)} \times \frac{i\omega}{(i\omega + \tau(\omega_0^2 - \omega^2))}$$ replace $i\omega \rightarrow s$

For $t > 0$ Laplace transform is equivalent. Also complete the square.

$$V(s) = \frac{q_0 R}{\tau\tau_s}\left(\frac{1}{s + 1/\tau_s}\right)\left(\frac{s}{(s + \frac{1}{2\tau})^2 + (\omega_0^2 - \frac{1}{4\tau^2})}\right)$$

set $\omega'^2 = (\omega_0^2 - \frac{1}{4\tau^2})$

pmt pulse with inductance.

The next two slides can be skipped if too detailed. But the point is that an inductance creates sinusoidal oscillations in the output pulse. This can be analyzed.

$$V(s) = \frac{q_0 R}{\tau\tau_s}\left(\frac{1}{s+\frac{1}{\tau_s}}\right)\left(\frac{s}{(s+\frac{1}{2\tau})^2+\omega'^2}\right)$$

Determine the poles of this formula

First we have to take this apart by partial fractions. This is going to get complicated, but doable.

$$\mathrm{V(s)} = \frac{q_0 R}{\tau\tau_s}\left(-A\frac{1}{s+\frac{1}{\tau_s}} + B\frac{1}{(s+\frac{1}{2\tau})^2+\omega'^2} + A\frac{(s+\frac{1}{2\tau})}{(s+\frac{1}{2\tau})^2+\omega'^2}\right)$$

The terms are put into canonical forms that can be compared to Laplace transforms in a handbook. The answer to an inverse transform is as follows.

$$v(t) = \frac{q_0 R}{\tau\tau_s}\left(-A(e^{-t/\tau_s} - e^{-t/2\tau}\,\mathrm{Cos}(\omega' t)) + Be^{-t/2\tau}\,\mathrm{Sin}(\omega' t)\right)$$

$$A = \frac{1}{\tau_s}\left[\frac{1}{(1/\tau_s - 1/2\tau)^2+\omega'^2}\right]$$

$$B = \frac{(1/2\tau)^2 - (1/2\tau\tau_s) + \omega'^2}{((1/\tau_s - 1/2\tau)^2 + \omega'^2)\omega'}$$

Both A and B have units of time.

Notice that when $\omega_0 \to 0$, ω' becomes imaginary, and the oscillatory terms turn into exponentials that go back to the solution without inductance.

table and plots for pmt with inductance

Set some parameters

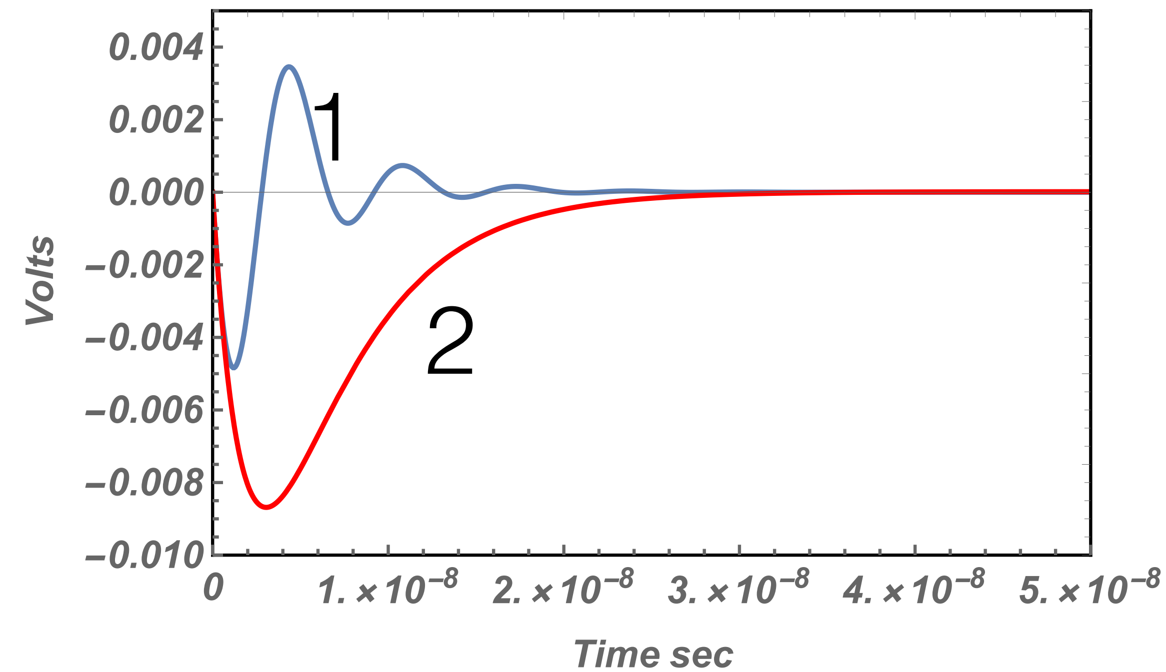
$q = 1.6 \times 10^{-12}\,C$

$R = 50\Omega$

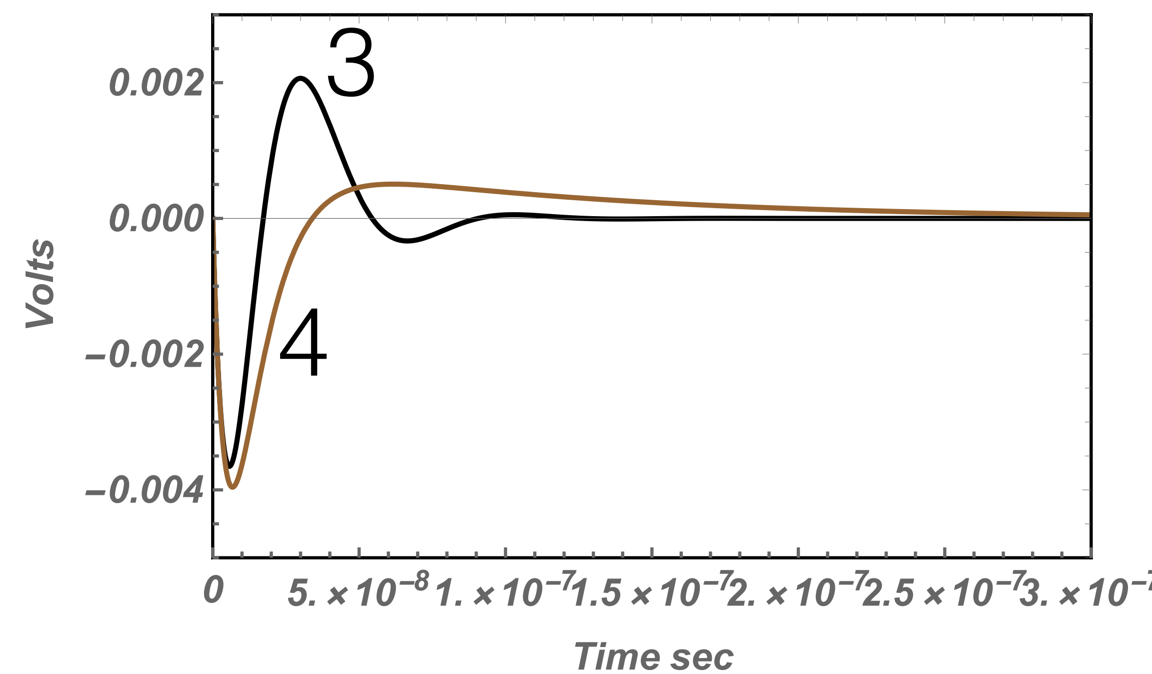
$\tau_s = 5 \times 10^{-9}$ sec

we are going to let $\tau = RC$ and $\omega_0 = 1/\sqrt{LC}$ vary

	tau ns	w0 Mhz	w’ Mhz	A/ts	B/ts
1	2	1 Ghz	968	0.042	0.208
2	2	10 MHz	i*250	-0.67	-0.67*i
3	10	100 MHz	86.6	1.33	0
4	10	30 MHz	i *40	1.91	i*2.18

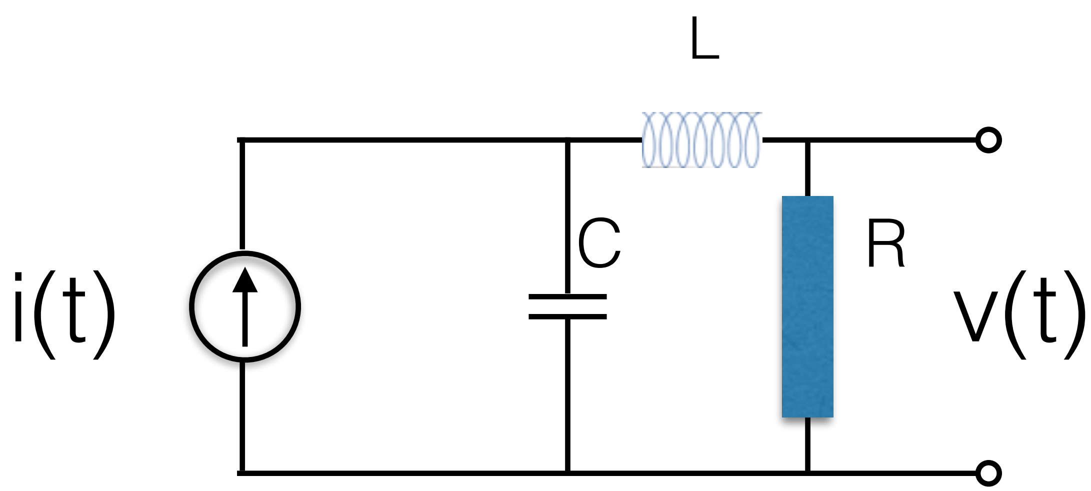


in all cases there is a positive tail



Effects of inductance at the output can cause oscillations or overshoots if the characteristic oscillation timescale Sqrt[LC] is close to the RC time constant of the PMT

example 2



This might be a more likely way the network becomes.

Notice that R and L are in series. And V_0 is across R.

$$Z = \frac{1}{(1/(R+i\omega L)+i\omega C)} \times \frac{R}{(R+i\omega L)}$$

$$I(\omega) = V(\omega)/Z \Rightarrow V(\omega) = \frac{q_0}{(1+i\omega\tau_s)} \times \frac{R}{(1+i\omega RC-\omega^2 LC)}$$

Set $\tau = RC$ and $\omega_0^2 = 1/LC$

$$V(\omega) = \frac{q_0 R}{(1+i\omega\tau_s)} \times \frac{1}{(1+i\omega\tau-\omega^2/\omega_0^2))} \quad \text{replace } i\omega \to s$$

For $t > 0$ Laplace transform is equivalent. Also complete the square.

$$V(s) = \frac{q_0 R\omega_0^2}{\tau}\left(\frac{1}{s+1/\tau_s}\right)\left(\frac{1}{(s+\omega_0^2\tau/2)^2+\omega_0^2(1-\omega_0^2\tau^2/4)}\right)$$

set $\omega'^2 = \omega_0^2(1-\omega_0^2\tau^2/4)$

solution inductance in series

We will solve this another way that might be more straight-forward.

$$V(s) = \frac{q_0 R\omega_0^2}{\tau}\left(\frac{1}{s+1/\tau_s}\right)\left(\frac{1}{s+1/\tau_e+i\omega'}\right)\left(\frac{1}{s+1/\tau_e-i\omega'}\right)$$

$$1/\tau_e = \omega_0^2\tau/2,\ \omega' = \omega_0(1-\omega_0^2\tau^2/4)^{1/2}$$

This can be easily solved by partial fractions to show

$$v(t) \ = \ \frac{q_0 R\omega_0^2}{\tau}\times\left[\frac{e^{-t/\tau_s}}{(1/\tau_s-1/\tau_e)^2-\omega'^2}+\frac{ie^{-t(1/\tau_e+i\omega')}}{2\omega'(1/\tau_s-1/\tau_e-i\omega')}-\frac{ie^{-t(1/\tau_e-i\omega')}}{2\omega'(1/\tau_s-1/\tau_e+i\omega')}\right]$$

This can be easily seen to be a real function of exponentials and sinusoidals.

If ω' is not real then one ends up with just exponentials.

Based on this we can conclude that a general function such as

$$v(t) \ = \ \mathrm{A}e^{-t/\tau_1} + Be^{-t/\tau_2}\,\mathrm{Sin}(\omega t) + Ce^{-t/\tau_2}\,\mathrm{Cos}(\omega t)$$

would work well to fit a pulse from a phototube connected to unknown impedances.

The relative values and signs of A, B, and C determine the bipolar shape of the pulse.

Next slide has example plots from above

table and plots for pmt with inductance in series

Set some parameters

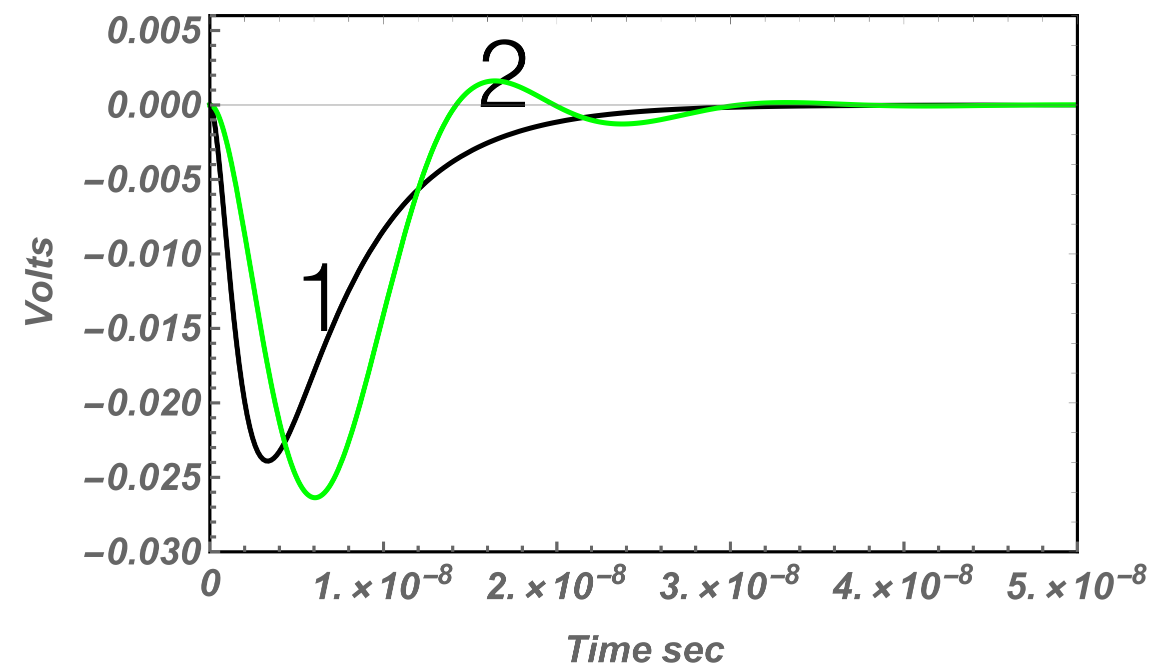
$q = 1.6 \times 10^{-12} C$

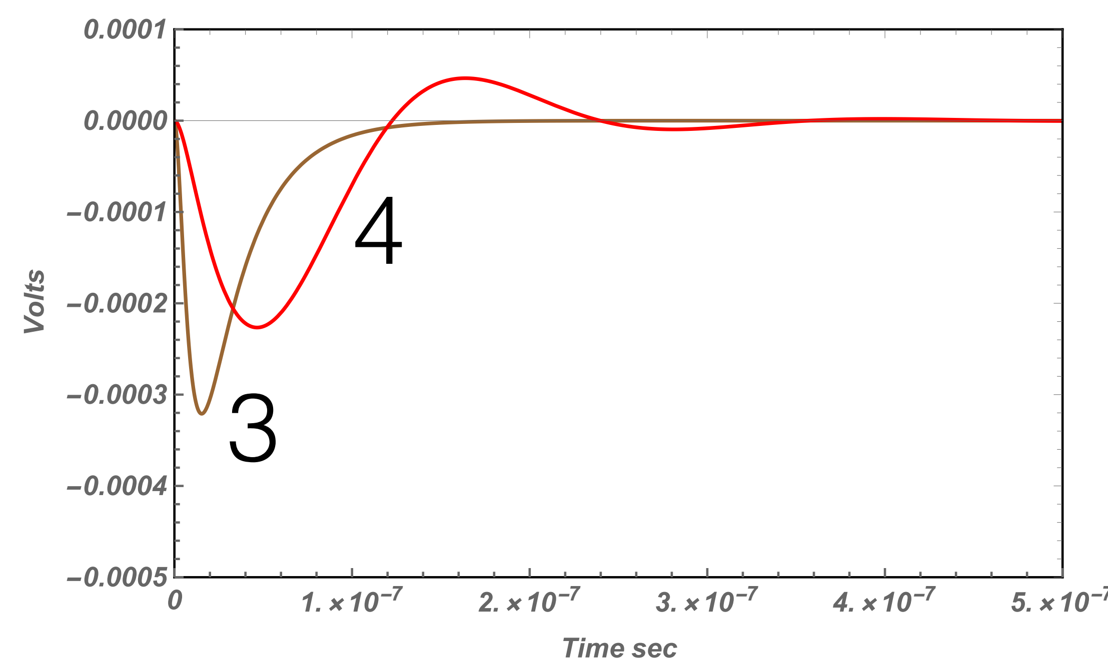
$R = 50\Omega$

$\tau_s = 5 \times 10^{-9}$ sec

we are going to let $\tau = RC$ and $\omega_0 = 1/\sqrt{LC}$ vary

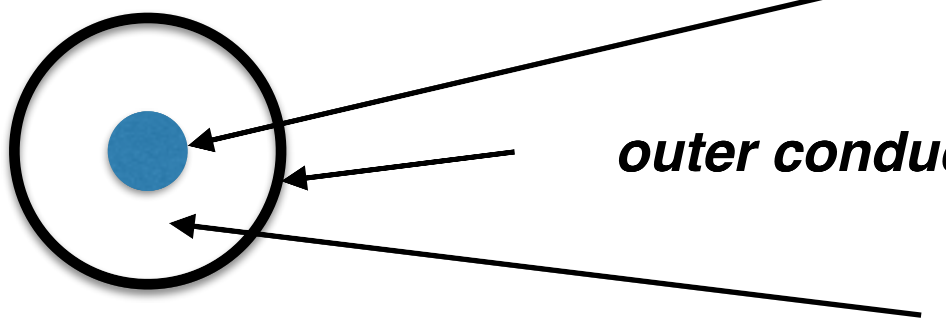
	tau ns	w0 Mhz	w' Mhz
1	2	1 Ghz	0
2	2	400 MHz	366
3	30	100 MHz	112*i
4	30	30 MHz	26.8





Effects of inductance at the output can cause oscillations or overshoots if the characteristic oscillation timescale Sqrt[LC] is close to the RC time constant of the PMT

Coaxial Cable



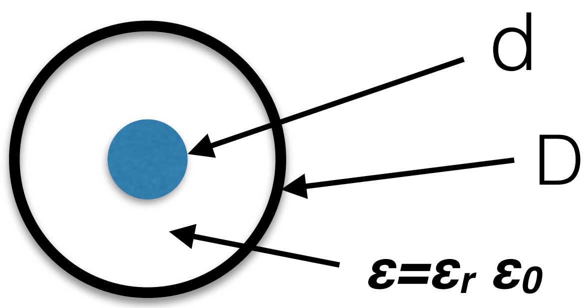
We will do some elementary calculations first, and then perform a more detailed analysis.

The idea is to use these notes as reference when needed.

Recall that the same treatment can be used for any pair of conductors that are parallel to each other and are used to transmit an electrical signal.

Signal gets transmitted by alternating electric and magnetic fields between the two conductors.

Capacitance of coaxial cable



Assume a charge of λ per unit length on the inner conductor. Then for length dl the total charge is $q = \lambda dl$

The electrical field at radius $d < r < D$ is radial and given by

$$2\pi r dl E_r \varepsilon_0 \varepsilon_r = q = \lambda dl$$

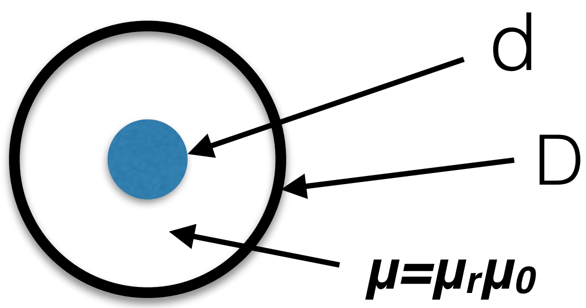
$$E_r = \frac{\lambda}{2\pi r \varepsilon_0 \varepsilon_r}$$

Voltage difference

$$\Delta V = \frac{\lambda}{2\pi\varepsilon_0\varepsilon_r}\int_d^D \frac{1}{r}dr = \frac{\lambda}{2\pi\varepsilon_0\varepsilon_r}\log(D/d)$$

Capacitance per unit length $c = \dfrac{\lambda}{\Delta V} = \dfrac{2\pi\varepsilon_0\varepsilon_r}{\log(D/d)}$

Inductance of coaxial cable

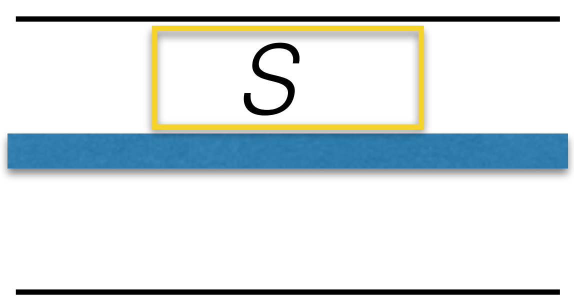


Assume there is current I in the inner conductor and $-I$ in outer.

The magnetic field in the cable for $d < r < D$ is given by.

$$2\pi r B = \mu_0 \mu_r I$$

$$B = \frac{\mu_0 \mu_r I}{2\pi r}$$



we take a rectangular surface area S (length ι) normal to the B field which is circular around the central conductor.

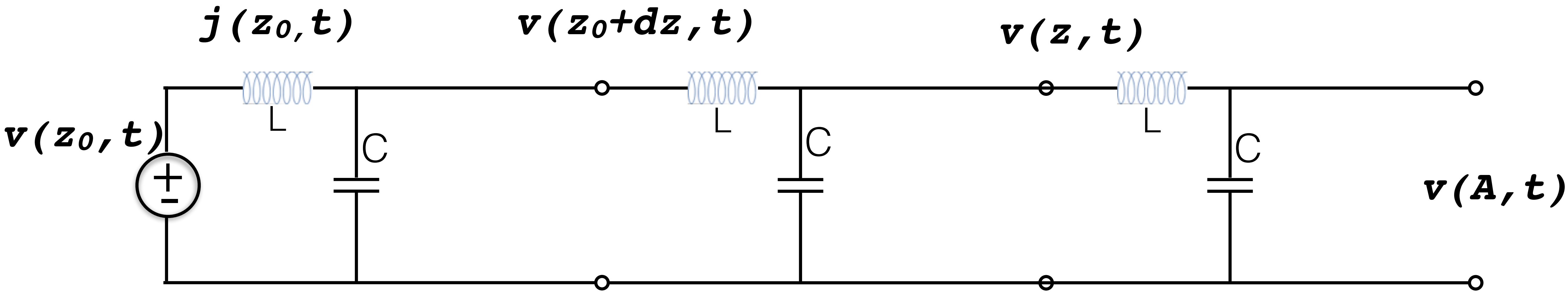
$$\phi = \int_S B\,ds = l \int_d^D \frac{\mu_0 \mu_r I}{2\pi r} dr$$

Inductance per unit length = $\dfrac{\phi}{\iota * I} = \dfrac{\mu_0 \mu_r}{2\pi} \log(D/d)$

Recall some numbers

$$\mathrm{c} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

$\mu_0 = 4\pi\ 10^{-7} H/m$ and so $\varepsilon_0 = 1/(\mu_0 c^2) \approx 8.85\ 10^{-12} F/m$



Cable is of length A, and each section has inductance $L = \ell dz$

and capacitance $C = cdz$. Now we get two coupled equations. $j(z,t)$ is the current at z, and t.

$$v(z+dz,t) - v(z,t) = -\ell dz \frac{dj(z,t)}{dt}$$ This has to do with the voltage drop across the inductor.

$$j(z+dz,t) - j(z,t) = -cdz \frac{dv(z,t)}{dt}$$ This is the current going to ground through the capacitor

$$\frac{dv(z,t)}{dz} = -\ell \frac{dj(z,t)}{dt}$$

$$\frac{dj(z,t)}{dz} = -c\frac{dv(z,t)}{dt} \qquad \Rightarrow \qquad \frac{d^2v(z,t)}{dz^2} = \ell c \frac{d^2v(z,t)}{dt^2}$$

We are going to use a double Fourier transform for the variables z,t. The conjugate variable for length is the wave number, k and for time it is frequency ω.

$$v(z,t) = \int V(k,\omega)e^{+i\omega t}e^{-ikz}dkd\omega; \quad j(z,t) = \int J(k,\omega)e^{+i\omega t}e^{-ikz}dkd\omega$$

Cable transmission (lossless) example

Plugging into the wave equation (this is true only when ω and k are not 0)

$$-k^2 V(k,\omega) = -\ell c \omega^2 V(k,\omega)$$

The condition of the wave equation is that

$$k^2 = \ell c \omega^2 \quad \text{or} \quad \beta = \omega / k = 1/\sqrt{\ell c}$$

The velocity of the wave through the cable is given by $1/\sqrt{\ell c}$. Going back to the coupled equations.

$$ikV(k,\omega) = i\omega\ell J(k,\omega)$$

This gives us the cable impedance which is real.

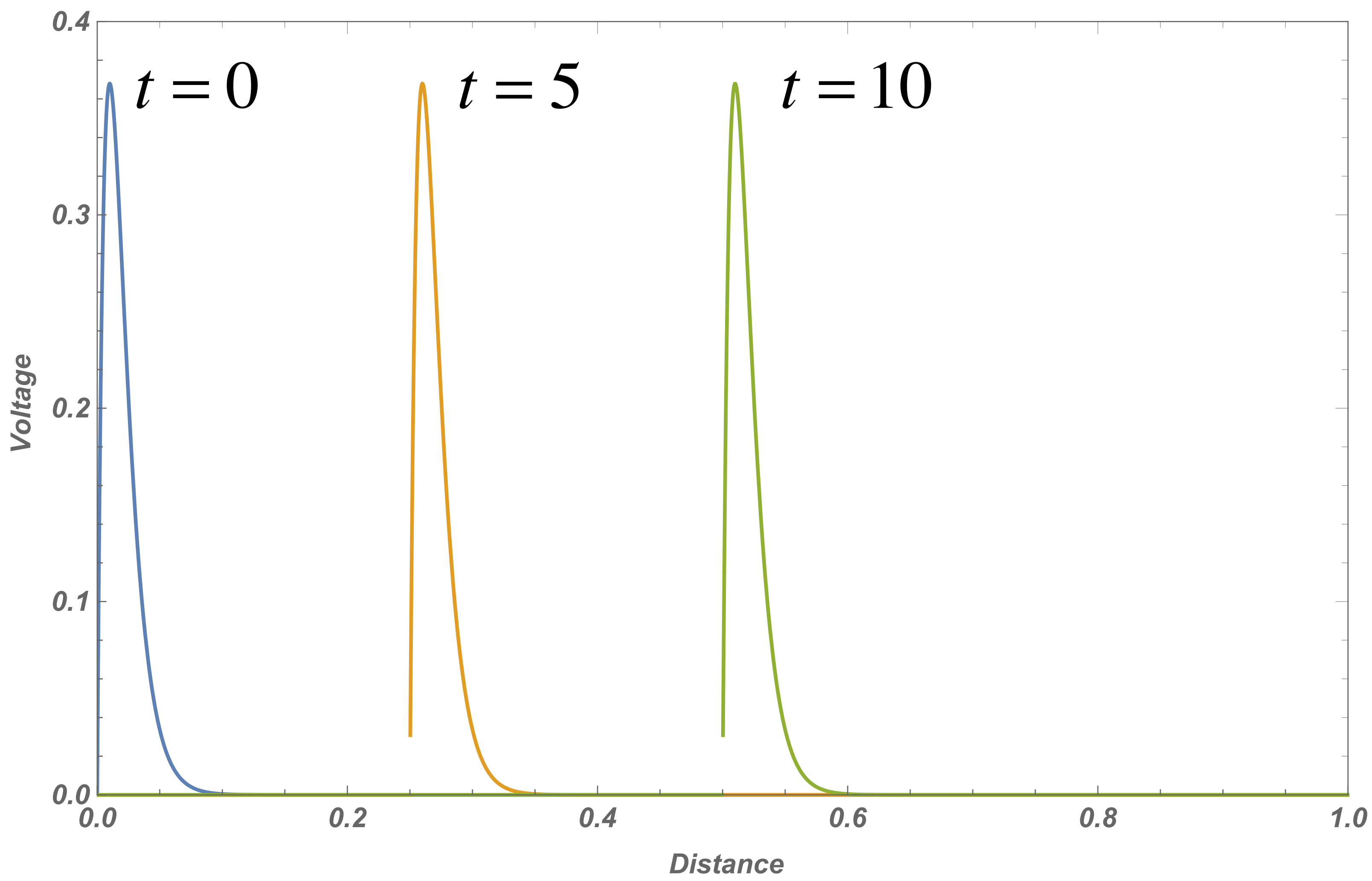
$$Z_0 = \frac{V(k,\omega)}{J(k,\omega)} = \ell\frac{\omega}{k} = \sqrt{\frac{\ell}{c}} \quad \text{.....}$$ We now use the formulas for ℓ and c.

$$\beta = 1/\sqrt{\ell c} = 1/\sqrt{\frac{\mu_0\mu_r}{2\pi}\log(D/d)\frac{2\pi\varepsilon_0\varepsilon_r}{\log(D/d)}} = \frac{c_{light}}{\sqrt{\mu_r\varepsilon_r}}$$ ... only depends on dielectric.

$$Z_0 = \sqrt{\frac{\frac{\mu_0\mu_r}{2\pi}\log(D/d)}{\frac{2\pi\varepsilon_0\varepsilon_r}{\log(D/d)}}} = \frac{\log(D/d)}{2\pi}\sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}} = \frac{\log(D/d)}{2\pi}\mu_0 c_{light}\sqrt{\frac{\mu_r}{\varepsilon_r}}$$

$$Z_0 = 60\Omega \times \sqrt{\frac{\mu_r}{\varepsilon_r}}\log(\frac{D}{d})$$

Graphical Solution



$k = 100$

$\omega = 5$

$v = \omega / k = 0.05$

$CableLeng = 1.0$

The solution to the wave equation is a continuous function with the following form

$v(z,t) = f(kz \pm \omega t)$

The sign of k determines the direction of the pulse movement.

Solution

We can also think about the solution in Laplace or Fourier space (also known as Phaser space)

$v(z,t) = f(kz \pm \omega t)$

Notice that this could be considered a function with a time shift $t \to t \pm z/\upsilon$

The time shift is z dependent. i.e. given a location along the cable, the signal is time shifted by $\pm z/\upsilon$. In Fourier space this is just a phase shift.

On page 8 we obtained the solution to a PMT pulse. In case of a cable attached to the PMT with a velocity constant υ, the solution at z is

$$V(\omega) \to V(\omega) \times e^{-i\omega(z/\upsilon)} = \frac{q_0 R}{(1+i\omega\tau_s)(1+i\omega\tau)} e^{-i\omega(z/\upsilon)}$$

This is going to be very useful. We can call the phase factor the propagator along the cable. If a cable is of length l, then every time the signal goes from one end to the other it gets a phase factor $\mathrm{e}^{-i\omega(l/\upsilon)}$

Some notes

- ***It is useful to understand the cable in terms of some analogies.***
- ***The inductance represents the tendency of the cable to oppose a change in current. It slows down the movement of charge down a cable.***
- ***The capacitance can be thought of as the tendency of the cable to oppose a change in voltage. The characteristic impedance is the balance of the L and C.***
- ***Any shunt resistances (through the dielectric) along the cable act as dissipators of the charge, somewhat like leaks in a pipe.***
- ***As the charge flows through the inductor (if the rate is increased too fast it is slowed), it fills the capacitor until the voltage increases so that the charge can flow forward again.***
- ***For an infinite cable, if the voltage is turned on suddenly at one end, the current flows continuously filling up the capacitors. Therefore an infinite cable is the same as a resistor.***
- ***As the current reaches the end of the cable, if it sees a resistor of the same value as the cable impedance then it is as if the cable is infinite. This is called proper cable termination.***
- ***The impedance of the cable is finite only for high frequencies or small wavelength (compared to its length).***

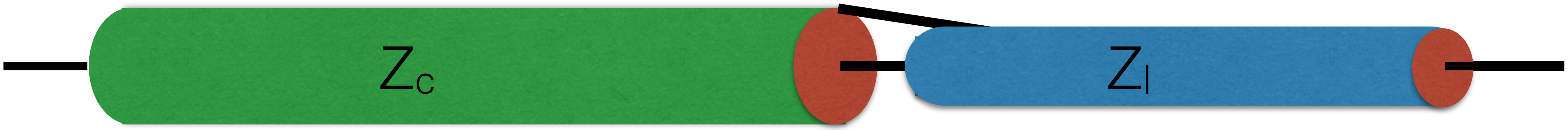
Imperfect termination

If the cable is shorted, charge traveling on the center will return on the shield. A voltage pulse will return with reversed polarity.

If the cable is terminated correctly charge traveling on the center will dissipate in the load. A voltage pulse will not return.

If the cable is open, charge traveling on the center will return on the center. A voltage pulse will return with same polarity.

Impedance matching



Suppose there are two infinite cables attached in the middle.

cable 1 has impedance Z_c and cable 2 impedance Z_l

Cable 2 can also be thought of as the load (or termination for cable 1).

At the boundary, there is a current incident from the left: $i_i(t)$

There is a reflected current: $i_r(t)$; There is a transmitted current: $i_t(t)$

$i_i(t)+i_r(t)=i_t(t)$

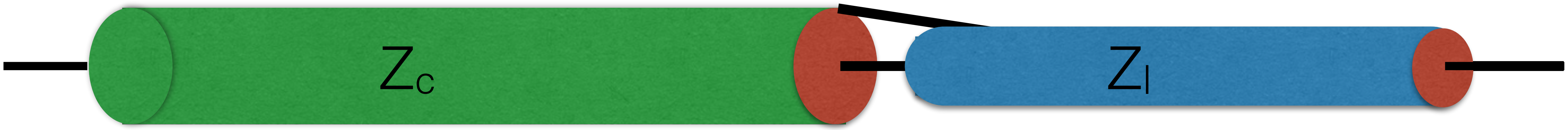
The cable impedance, in general, is complex and frequency dependent, and so it is better to use the Fourier transforms of the currents and voltages.

$I_i + I_r = I_t \qquad V_i + V_r = V_t \quad$ and $\; V_i = Z_c I_i \qquad V_r = -Z_c I_r \qquad V_t = Z_l I_t$

$$\Rightarrow \; I_i - I_r = \frac{Z_l}{Z_c} I_t$$

The sign of the voltage for reflected current is important. Secondly, imagine that cable 2 is the input to a scope. Then the scope is going to measure V_t.

Reflection and transmission



$$I_i \Rightarrow \qquad\qquad I_t = I_i \times \frac{2Z_c}{Z_l + Z_c} \Rightarrow$$

$$I_r = I_t \times \frac{Z_c - Z_l}{2Z_c} = I_i \frac{Z_c - Z_l}{Z_c + Z_l} \Leftarrow$$

$$V_r = V_i \frac{Z_l - Z_c}{Z_c + Z_l} \qquad\qquad V_t = V_i \times \frac{2Z_l}{Z_c + Z_l} = T \times V_i$$

As $Z_l \to \infty$, the reflection $V_r \to V_i$ $\quad V_t \to 2V_i$

But as $Z_l \to 0$, The reflection $V_r \to -V_i$ $\quad V_t \to 0$

For ease of use we are going to use $\Lambda_l = \dfrac{Z_l - Z_c}{Z_c + Z_l}$

If there is a termination at the source then there will be reflections from the source end when a reflected pulses reaches: $\Lambda_s = \dfrac{Z_s - Z_c}{Z_c + Z_s}$ We will use this later.

Pulse train

Imagine that at the source I start $V(\omega)$, what will I measure across the load ?

$V(\omega) \to V(\omega)e^{-i\omega(l/v)} \to V(\omega)e^{-i\omega(l/v)}T$ ... This is the first pulse T$=\dfrac{2Z_l}{Z_l + Z_c}$

$V(\omega)e^{-i\omega(l/v)}\Lambda_l \to V(\omega)e^{-3i\omega(l/v)}\Lambda_l\Lambda_s T$.......This is the second pulse

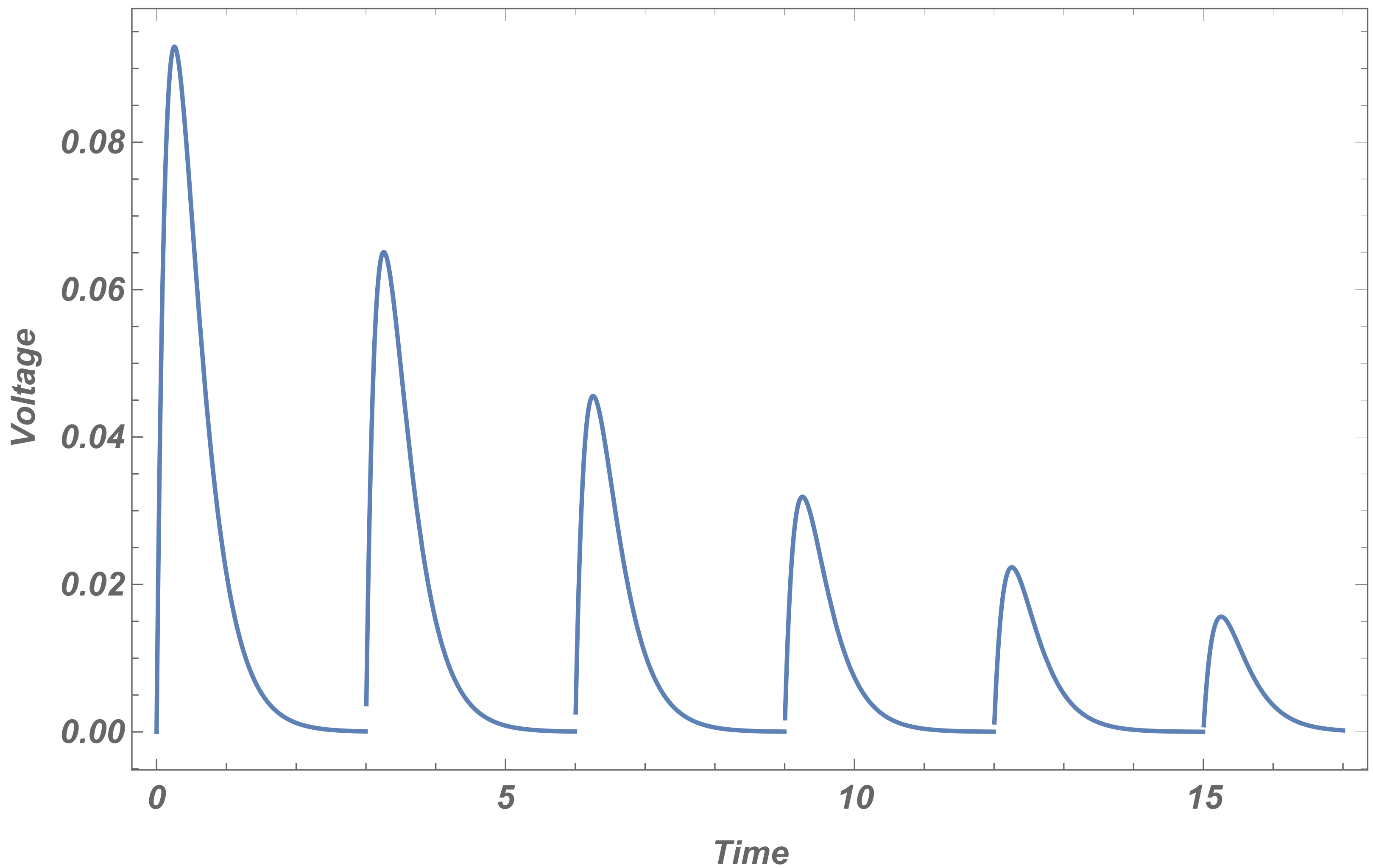
$V(\omega)e^{-3i\omega(l/v)}\Lambda_l^2\Lambda_s \to V(\omega)e^{-5i\omega(l/v)}\Lambda_l^2\Lambda_s^2 T$.....This is the third pulse

$\Rightarrow V(\omega)Te^{-i\omega(l/v)}\left(\sum_{n=0}^{\infty}\left[e^{-2i\omega(l/v)}\Lambda_l\Lambda_s\right]^n\right)$ .... This is a geometric series.

$$=V(\omega)Te^{-i\omega(l/v)}\frac{1}{1-e^{-2i\omega(l/v)}\Lambda_l\Lambda_s}$$

$$v(t) = \int_{-\infty}^{\infty} V(\omega)Te^{-i\omega(l/v)}\frac{1}{1-e^{-2i\omega(l/v)}\Lambda_l\Lambda_s}e^{+i\omega t}\frac{d\omega}{2\pi}$$

The solution is an inverse Fourier transform of the above. If $(\Lambda_l\Lambda_s < 1)$ is real then the pulse shape is preserved and it just gets smaller and smaller for each subsequent reflection. However, the source and load impedances may depend on ω, then there will be a distortion on the pulse as it bounces back nd forth.



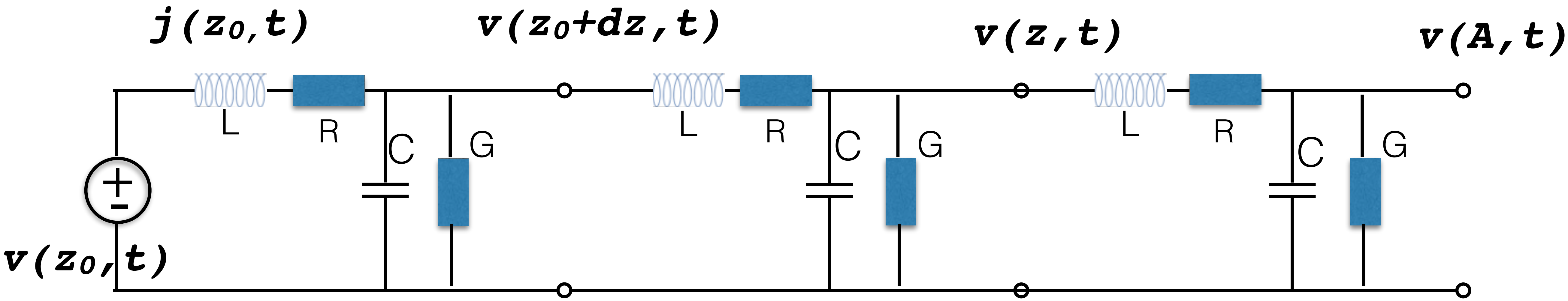
$\tau_s = 1/5$

$\tau_l = 1/3$

$T = 3$ (cable length in time)

$\Lambda_l \Lambda_s = 0.7$

Cable dispersion



Cable is of length A, and each section has inductance $L = \ell dz$ and series resistance $R = rdz$ and capacitance $C = cdz$ and conductance $G = gdz$. Two coupled equations are now complicated.

$$\frac{dv(z,t)}{dz} = -\ell \frac{dj(z,t)}{dt} - r\ j(z,t)$$

$$\frac{dj(z,t)}{dz} = -c\frac{dv(z,t)}{dt} - g\ v(z,t) \qquad \Rightarrow \text{ This leads to the telegraph equation.}$$

$$\frac{d^2v(z,t)}{dz^2} = \ell c\frac{d^2v(z,t)}{dt^2} + \ell g\frac{dv(z,t)}{dt} + rc\frac{dv(z,t)}{dt} + rg\, v(z,t)$$

$$\frac{d^2j(z,t)}{dz^2} = \ell c\frac{d^2j(z,t)}{dt^2} + \ell g\frac{dj(z,t)}{dt} + rc\frac{dj(z,t)}{dt} + rg\, j(z,t)$$

This is still a linear system (output doubles if input doubles) and therefore it can be solved using Fourier or Laplace analysis. But to implement it fully requires a numerical calculation.

General formula for impedance

We again start with the definition

$$v(z,t) = \int V(k,\omega) e^{+i\omega t} e^{-ikz} dk d\omega; \quad j(z,t) = \int J(k,\omega) e^{+i\omega t} e^{-ikz} dk d\omega$$

Then

$$-ikV(k,\omega) = -rJ(k,\omega) - i\omega\ell J(k,\omega)$$

$$-ikJ(k,\omega) = -gV(k,\omega) - i\omega c V(k,\omega)$$

This gives

$$Z = \frac{V}{J} = \sqrt{\frac{r + i\omega l}{g + i\omega c}}$$

care needed: g is conductance per unit length.

In the limit of $r,\ g \to\ 0$, we get back the usual formula $Z_0 = \sqrt{\ell / c}$

For most practical dielectrics g is very small, and r is the dominant factor.

But even r tends to be small compared to $\omega\ell$.

For most practical cables, just using Z_0 as the impedance to determine what happens at the termination is sufficient. However, generally only a small phase shift can be expected at the termination point.

The series resistance r is frequency dependent and increases with frequency. We will calculate this.

Telegraph equation

$$\frac{d^2 v(z,t)}{dz^2} = \ell c \frac{d^2 v(z,t)}{dt^2} + \ell g \frac{dv(z,t)}{dt} + rc \frac{dv(z,t)}{dt} + rg\, v(z,t)$$

$$-k^2 = -\ell c \omega^2 - i\omega(\ell g + rc) + rg$$

Set $\beta^2 = 1/\ell c$ This is the velocity when there is no dissipation.

$$\beta^2 k^2 = (\omega - \omega_1)^2 + \omega_2^2$$

$$\omega_1 = i\frac{(g/c + r/\ell)}{2} \quad \text{and} \quad \omega_2^2 = \frac{g^2}{4c^2} + \frac{r^2}{4\ell^2} - \frac{rg}{2\ell c}$$

If $g/c \ll \omega$ and $r/\ell \ll \omega$ we can ignore ω_2 to first order.

Now we try to get the propogation constant for this wave

$$v = \omega / k \approx \frac{\omega\beta}{(\omega - \omega_1)} \approx \beta(1 + \frac{\omega_1}{\omega}).....$$ remember that ω_1 is imaginary with units of 1/time,

and so it will diminish the wave. The propagation factor is $e^{-i\omega(l/v)}$

Phase factor for dissipative wave is $\sim \omega(l/\beta)(1 - \omega_1/\omega)$

$$\Rightarrow e^{-i\omega(l/\beta)} \times e^{-(l/\beta)\times(g/c + r/\ell)/2}$$

To first order the wave simply gets exponentially reduced over all frequencies with

a propagation constant of $\gamma = \frac{(gZ_0 + r/Z_0)}{2}$

Cable Attenuation due to skin effect

$\gamma = \frac{(gZ_0 + r/Z_0)}{2}$ has units of 1/distance. This means it is reciprocal of the attenuation length.

For excellent dielectrics there should be no loss of current, and therefore we will set $g \approx 0$.

The series resistance (r) is a function of frequency because of the skin effect. We will derive the skin effect in another series of notes, but here we take the simple approximation.

The approximation states that for frequency ω, all of the current in a conductor is at the surface within depth of $\delta_s = \sqrt{\frac{2\rho}{\omega(\mu_R\mu_0)}}$ where ρ is resistivity and $\mu = \mu_R\mu_0$ is magnetic permeability.

For copper: $\rho = 1.68 \times 10^{-8}\,\Omega.m$ and $\mu \approx \mu_0 = 4\pi \times 10^{-7}\,H/m$

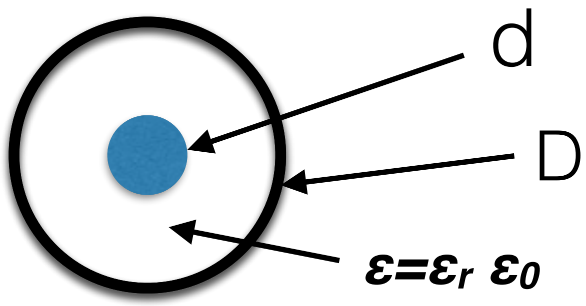
Engineers prefer to use cycle frequency $f = \omega/2\pi$

Notice that the skin depth is <<1 mm for frequencies above MHz. In these cases we can ignore the thickness of the wire and just calculate the surface volume to calculate the resistance at that frequency

Also notice that for high frequencies one could make wires that just have excellent surface conductors. We are going to assume solid copper, however.

f	delta(micron)	delta(mm)
60 Hz	8421	8.4
1 kHz	2062	2.1
10kHz	652	0.65
100kHz	206	0.21
1MHz	65	0.065
10Mhz	21	0.021
100Mhz	6.5	0.0065

Coaxial cable series resistance



We use the surfaces of the inner and outer conductors and put them in series to get the total resistance per unit length:

$$r = \rho\left(\frac{1}{\pi d\delta} + \frac{1}{\pi D\delta}\right)$$

Now we use this to calculate the propagation constant

$$\gamma = \frac{1}{2Z_0}\left[\frac{\rho}{\pi d\delta} + \frac{\rho}{\pi D\delta}\right] = \frac{1}{2\sqrt{2}\pi Z_0}\sqrt{\omega\mu_R\mu_0\rho}\left[\frac{1}{d} + \frac{1}{D}\right]$$

We can now calculate this for various types of cables. Most important: attenuation increases as $\sqrt{\omega}$ or the attenuation length decreases with sqr-root of frequency.

cable	Z0 (Ohm)	beta/c (velocity)	C (pF/m)	core dia (mm)	Shield dia (mm)	1/gam (m)	DB/100ft @100Mhz
RG6/U	75	0.68	65.6	1.024	4.7	102	2.6(2.7)
RG8/U	50	0.69	96.8	2.17	7.2	203	1.3(1.9)
RG58/U	50	0.71	93.5	0.81	2.9	77	3.4(4.6)
RG59/U	73	0.659	70.5	0.644	3.71	97.7	2.7(3.4)
RG174/U	50	0.68	98.4	7x0.16	1.5	32	8.2(8.9)

Commonly used cables. RG6 is the TV cable. RG174 has 7 strands, we assume effective diameter of burden to be ~2 times the diameter of a strand. The velocity and attenuation are calculated. They are slightly different from the specs which are in brackets. (from Moore,Davis,Coplan)

What is DB ?

Let's do some cleanup here.

First attenuation is given by $A(\omega)=e^{-l\times\gamma}=e^{-l\cdot\left[\frac{1}{2\sqrt{2}\pi Z_0}\sqrt{\mu_R\mu_0\rho}\left[\frac{1}{d}+\frac{1}{D}\right]\right]\sqrt{\omega}}$

*E*ngineers like to use DB scale which is $\mathrm{A}_{db}=\text{-}20\times\mathrm{Log}_{10}(\mathrm{A})$

$$\frac{\mathrm{A}_{db}}{l}=\frac{20}{Ln[10]}\left[\frac{1}{2\sqrt{\pi}Z_0}\sqrt{\mu_R\mu_0\rho}\left[\frac{1}{d}+\frac{1}{D}\right]\right]\times 10^3\sqrt{f/Mhz}$$

$$\frac{\mathrm{A}_{db}}{l}=7.12\times 10^{-6}\left[\frac{1}{d}+\frac{1}{D}\right]\times\sqrt{f/Mhz}$$

The nice thing about this scale is that you just have to multiply by the length and Sqrt[f] to get the A_{db}

But for physicists it is totally confusing, and we are quite allowed to use the attenuation length at 1 Mhz as the quantity to remember.

We are now going to use this result to simulate what happens to a pulse over a length of cable. The assumptions are that only high frequencies matter, and that the cable is much longer that the longest wavelength in the signal.

For RG58/U $\mathrm{A}(\omega)=\mathrm{e}^{-l\sqrt{\omega}\times Const}$ $\mathrm{Const}=5.16\times 10^{-7}/(m\sqrt{Hz})$

For RG59/U $\mathrm{Const}=4.08\times 10^{-7}/(m\sqrt{Hz})$

Always remember $\omega=2\pi\mathrm{f}$ whenever using these formulas.

Example cable attenuation calculation

Use a discrete Fourier transform. Careful of the normalization and also the conventions of any particular computer program. Use definition: $\omega = 2\pi f$

We setup time scale T=2×10^{-7} sec and number of bins N=101.

Then δ=2×10^{-9} and $\delta f = 1/T$ and frequency range is $f_{range} = \frac{1}{T}\frac{(N-1)}{2}$

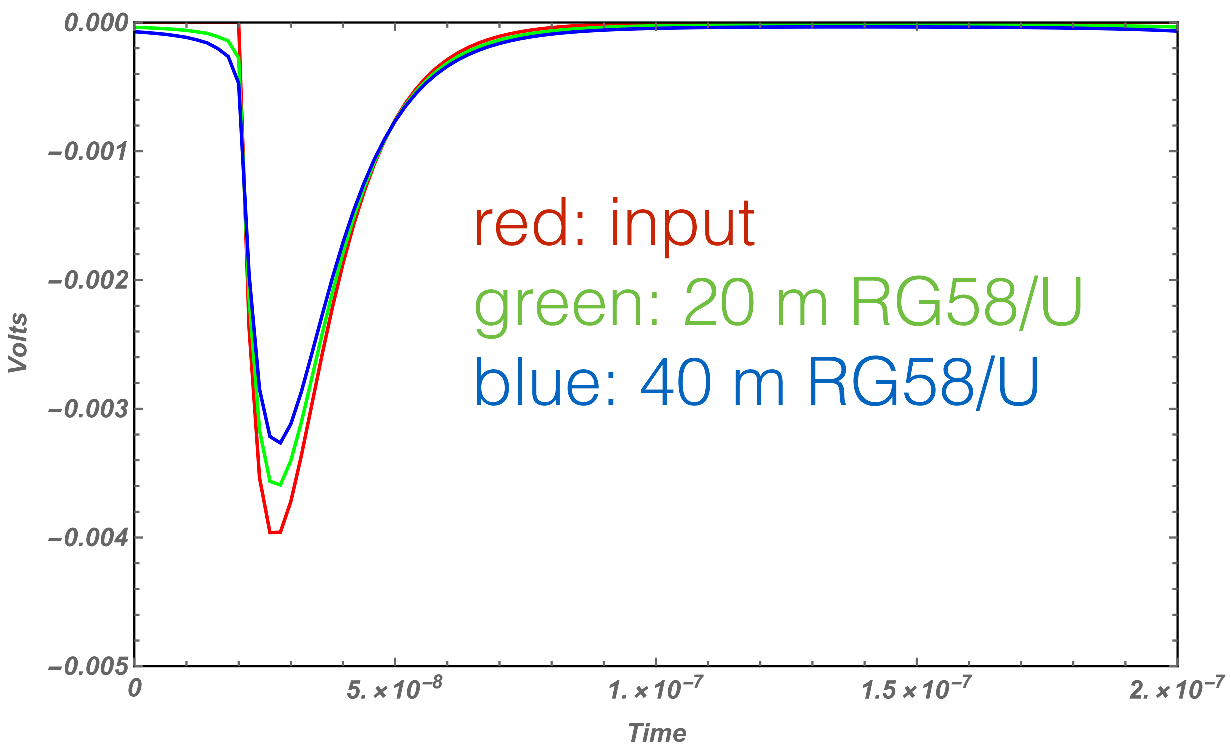
First we calculate $v(t_k) = -\frac{q_0 R}{(\tau-\tau_s)}\left(e^{-t/\tau_s} - e^{-t/\tau}\right)u(t)$ for $t_k = \delta\times k \quad k=0,N, \quad \tau_s = 5ns, \quad \tau = 10ns$

Then we take the DFT $V(\omega_j)$. $\omega_j = 0$ is usually the first element, and often

$V(\omega_j)$ needs to be shifted by $(N\text{-}1)/2$ bins to correspond to $\omega_j = 2\pi\delta f\times j \quad j = -\frac{(N-1)}{2}, +\frac{(N-1)}{2}$

We then reweight the $V_{a-shifted}(\omega_j) = V_{shifted}(\omega_j)\times e^{-5.16\times10^{-7}(L/m)\sqrt{Abs[\omega_j]}}$ for cable length L.

We may have to shift the $V_{a-shifted}$ back to $V_a(\omega_j)$ and take the inverse DFT.



I have shifted the waveform to display how an early and late tail develops when high frequencies are attenuated.

Notice the slight upturn at the end of the plot for long cable length. This is a consequence of the DFT which is N-periodic. The upturn actually belong on the early side of the waveform.

conclusions

- ***In these notes I have examined one of the most common detector configurations in particle and nuclear physics: a photomultiplier tube coupled to a long cable.***

- ***The PMT can be modeled as a current source with an impedance.***

- ***The cable transmission can be modeled over most of the frequencies as a phase shift and an attenuation that goes as Sqrt[frequency].***

- ***If there is an impedance mismatch, we can also model the effect as an infinite series with appropriate attenuation.***

- ***The set of equations introduced in these slides can be used for creating an accurate simulation of waveforms from the detector with tuning of a small set of parameters.***